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Inference for the Parameters of the Exponentiated Half Logistic Distribution

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Research Article

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Abstract

This study focuses on estimating the parameters of the Exponentiated Half-Logistic distribution through eleven distinct estimation techniques. These include maximum likelihood, least squares, weighted least squares, Cramér–von Mises, various forms of the Anderson–Darling method (standard, right-tailed, left-tailed, and second-order left-tailed), maximum product of spacings, minimum spacing absolute distance, and minimum spacing absolute-log distance estimators. A comprehensive Monte Carlo simulation is conducted to assess and compare the performance of these methods under varying parameter configurations and sample sizes. To illustrate the practical applicability of the proposed estimators, the corresponding distribution is fitted to a real-world wind speed data set. The results indicate that the exponentiated half logistic distribution provides a better fit than several commonly used models for this type of data.

Keywords: Exponentiated half logistic distribution, parameter estimation, Monte-Carlo simulation, model evaluation, wind speed

Üstel Yarı Lojistik Dağılımın Parametreleri İçin İstatistiksel Sonuç Çıkarım

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Öz

Bu çalışma, Üstel Yarı-Lojistik dağılımının parametrelerinin on bir farklı tahmin yöntemiyle belirlenmesine odaklanmaktadır. Kullanılan yöntemler arasında en çok olabilirlik, en küçük kareler, ağırlıklı en küçük kareler, Cramér–von Mises ve çeşitli Anderson–Darling istatistikleri (standart, sağ kuyruk, sol kuyruk ve ikinci mertebeden sol kuyruk), maksimum aralık çarpımı, minimum aralık mutlak uzaklığı ve minimum aralık logaritmik mutlak uzaklığı tahmin edicileri yer almaktadır. Bu yöntemlerin farklı parametre değerleri ve örneklem büyüklükleri altında gösterdiği performanslar, kapsamlı bir Monte Carlo benzetimi ile değerlendirilmiş ve karşılaştırılmıştır. Önerilen tahmin yöntemlerinin pratik uygulanabilirliğini göstermek amacıyla, üstelleştirilmiş yarı lojistik dağılım gerçek bir rüzgâr hızı veri setine uygulanmıştır. Elde edilen sonuçlar, bu dağılımın söz konusu veri türü için yaygın olarak kullanılan modellere kıyasla daha iyi bir uyum sağladığını ortaya koymaktadır.

Anahtar Kelimeler: Üstel yarı lojistik dağılım, parametre tahmini, Monte-Carlo benzetimi, model değerlendirmesi, rüzgar hızı

Introduction

The exponentiated half-logistic (EHL) distribution was introduced by Kang and Seo [1] as a flexible alternative to commonly used statistical distributions for modeling lifetime data. Moreover, it has been applied in many fields, including insurance, engineering, medicine, and education [2]. Kim et al. [3]

considered the Bayesian estimation of shape parameters, reliability function and failure rate functions when the underlying distribution is EHL under censored data. Kang et al. [4] obtained the maximum likelihood (ML) estimators of the unknown parameters of the EHL distribution based on a progressively Type-II censored sample. Seo and Kang [5] obtained the ML and moment estimators of the unknown parameters of the EHL distribution. Seo and Kang [6] proposed more efficient approaches for estimating the shape parameters in view of Bayesian and non-Bayesian perspective, for EHL record data. Gui [7] estimated the unknown shape and scale parameters of the EHL distribution by using the inverse moment and modified inverse moment estimators. Rao [8] considered the EHL distribution in the context of quality control by developing an attribute control chart for time truncated life tests with known or unknown shape parameters. Rao et al. [9] aimed to develop a multiple deferred state sampling plan for a time-limited life test, assuming that the lifetime of a product follows the EHL distribution. Alotaibi et al. [10] proposed a bivariate EHL distribution with univariate EHL marginals and examined various properties of the proposed distribution, such as the product moment, marginal, and conditional distributions. Rao et al. [11] developed a multiple dependent state repetitive sampling plan, assuming that the lifetime of a product follows the EHL distribution. Hassan and Morgan [12] considered the estimation of stress-strength reliability in the presence of outliers and homogenous cases when the stress and strength random variables follow EHL distribution.

The main contributions of this study can be summarized as follows:

i. This study provides a systematic overview of the two-parameter EHL distribution, focusing on the estimation of its shape and scale parameters using classical and distance-based methods.

ii. A comprehensive comparative framework is established by investigating eleven different parameter estimation methods for the EHL distribution, ML, least squares (LS), weighted least squares (WLS), Cramér-von Mises (CM), Anderson–Darling (AD), right-tailed Anderson–Darling (RTAD), left-tail Anderson-Darling (LTAD), Anderson-Darling left tail second order (ADLTS), maximum product of spacing (MPS), minimum spacing absolute distance (MSAD), and minimum spacing absolute log-distance (MSALD), which significantly extends the scope of existing studies that typically consider only a limited subset of estimators.

iii. Unlike previous works that mainly concentrate on ML, MPS, or Bayesian approaches, this study incorporates both goodness-of-fit–based estimators and minimum distance estimators, allowing for a more detailed evaluation of their performances.

iv. The efficiency and bias properties of all considered estimators are assessed through an extensive Monte Carlo simulation study using different sample sizes and parameter settings.

v. To the best of our knowledge, this study is the first to apply the EHL distribution for modeling wind speed data, thereby extending the applicability of the EHL distribution beyond traditional lifetime and reliability contexts.

vi. The practical performance of the estimation methods is further validated using a real data application, where goodness-of-fit tests are employed to determine the most effective estimator in an applied setting.

The remainder of this study is organized as follows. First, the EHL distribution, its distributional properties, the estimation methods, the Monte Carlo simulation study, and the real data application are presented. Finally, the main findings of the study and concluding remarks are provided.

Materials and Methods

We provide a brief introduction to the EHL distribution. We also consider the estimation of its parameters using various methods, including ML, LS, WLS, CM, AD, RTAD, LTAD, ADLTS, MPS, MSAD, and MSALD.

Exponentiated Half Logistic Distribution

Let X be a random variable following EHL distribution with shape parameter α and scale parameter β, respectively. The cumulative distribution function (cdf), probability density function (pdf), survival function (sf), and hazard rate function (hrf) are respectively given below:

$$F(x) = \left(\frac{1-e^{-\beta x}}{1+e^{-\beta x}}\right)^{\alpha}, x > 0, \alpha, \beta > 0, \tag{1}$$

$$f(x) = \frac{2\alpha\beta e^{-\beta x}}{(1+e^{-\beta x})^2}\left(\frac{1-e^{-\beta x}}{1+e^{-\beta x}}\right)^{\alpha-1}, x > 0, \alpha, \beta > 0, \tag{2}$$

$$S(x) = 1 - \left(\frac{1-e^{-\beta x}}{1+e^{-\beta x}}\right)^{\alpha}, x > 0, \alpha, \beta > 0, \tag{3}$$

$$h(x) = \frac{2\alpha\beta e^{-\beta x}F(x)}{(1-e^{-2\beta x})\big(1-F(x)\big)}, x > 0, \alpha, \beta > 0. \tag{4}$$

Figure 1 shows the graph of the pdf of the EHL distribution for various shape parameters. It can be seen from this figure that the pdf is decreasing for $\alpha \leq 1$, and, for $\alpha > 1$, it is a right-skewed unimodal function. As a special case, when $\alpha = 1$, the EHL distribution reduces to the half-logistic distribution [5]. Figure 2 presents the graph of the hrf of the EHL distribution for various shape parameters. According to Seo and Kang [5], for $\alpha < 1$, the hrf drops to a positive constant and then increases toward β. For $\alpha \geq 1$, the hrf increases toward β, exceeding the boundary of 1 for the shape parameters.

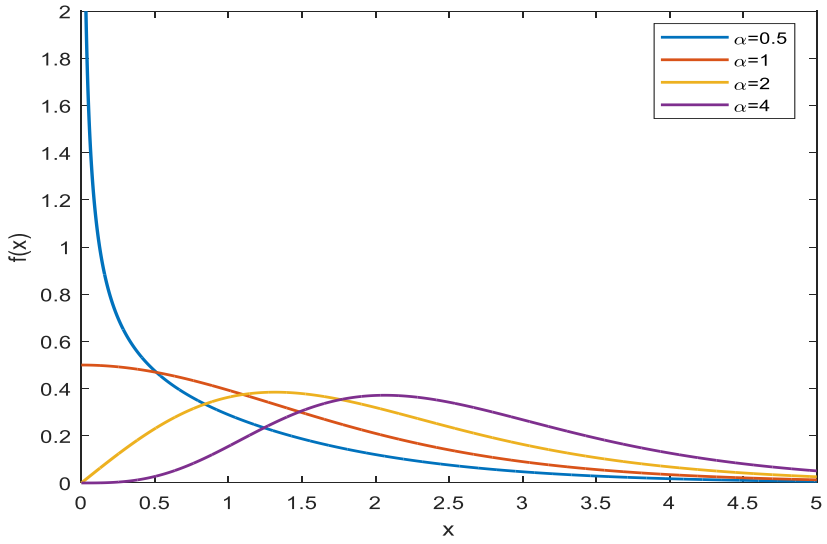


***Figure 1**. The pdf plots of the EHL distribution for different values of the shape parameters*

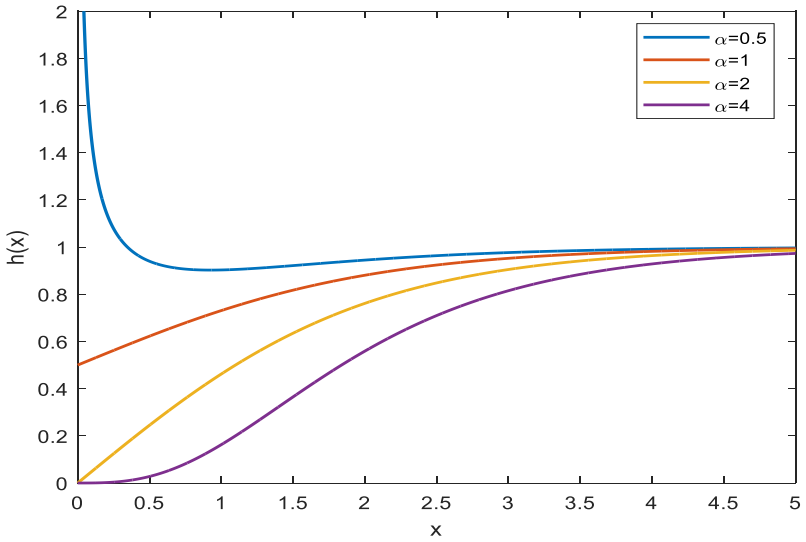


Figure 2. *The hrf plots of the EHL distribution for different values of the shape parameters*

Maximum Likelihood Estimators

Let $x_1, \dots, x_n$ be a random sample of size n drawn from the EHL distribution with parameters α and β. The likelihood function is given as follows:

$$L(\alpha, \beta | x_1, \dots, x_n) = \frac{2^n \alpha^n \beta^n e^{-\beta \sum_{i=1}^n x_i}}{\prod_{i=1}^n (1 + e^{-\beta x_i})^2} \prod_{i=1}^n \left(\frac{1 - e^{-\beta x_i}}{1 + e^{-\beta x_i}} \right)^{\alpha - 1}. \tag{5}$$

The log-likelihood function of the EHL distribution is given below:

$$\begin{aligned} \log L = n \log 2 \ + n \log \alpha + n \log \beta - \beta \sum_{i=1}^n x_i - 2 \sum_{i=1}^n \log(1 + e^{-\beta x_i}) \\ + (\alpha - 1) \sum_{i=1}^n \log \left(\frac{1 - e^{-\beta x_i}}{1 + e^{-\beta x_i}} \right). \end{aligned} \tag{6}$$

From (6), by taking derivatives of log-likelihood function with respect to α and β, the likelihood equations are obtained, as follows:

$$\frac{\partial}{\partial \alpha} \log L = \frac{n}{\alpha} + \sum_{i=1}^n \log \left(\frac{1 - e^{-\beta x_i}}{1 + e^{-\beta x_i}} \right) = 0, \tag{7}$$

$$\frac{\partial}{\partial \beta} \log L = \frac{n}{\beta} - \sum_{i=1}^n x_i + \sum_{i=1}^n \frac{2 x_i e^{-\beta x_i}}{1 + e^{-\beta x_i}} + (\alpha - 1) \sum_{i=1}^n \left(\frac{2 x_i e^{-\beta x_i}}{1 - e^{-2\beta x_i}} \right) = 0. \tag{8}$$

From (7), the ML estimator of α is obtained

$$\hat{\alpha}(\beta) = \frac{n}{\sum_{i=1}^n \log \left(\frac{1 + e^{-\beta x_i}}{1 - e^{-\beta x_i}} \right)}. \tag{9}$$

If equation (9) is substituted into (8), the ML estimator of β can be obtained as the solution of the following nonlinear equation

$$h(\beta) = \frac{n}{\beta} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{2x_i e^{-\beta x_i}}{1 + e^{-\beta x_i}} + \left(\frac{n}{\sum_{i=1}^{n} \log\left(\frac{1 + e^{-\beta x_i}}{1 - e^{-\beta x_i}}\right)} - 1 \right) \sum_{i=1}^{n} \frac{2x_i e^{-\beta x_i}}{1 - e^{-2\beta x_i}}. \quad (10)$$

It is observed that (10) cannot be solved explicitly. Therefore, numerical methods are employed for the ML estimator of β.

Least Squares Estimators

The LS estimators of the parameters are derived based on the approach proposed by Swain et al. [13]. Let $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ denote the ordered observations from a random sample of size n from EHL distribution. The LS estimators of α and β are obtained by minimizing the following expression with respect to the parameters of interest:

$$S = \sum_{i=1}^{n} \left(F\left(x_{(i)}\right) - h_i\right)^2. \quad (11)$$

Here, $h_i = \frac{i}{n+1}, i = 1, \ldots, n$ represents the expected value of $F\left(x_{(i)}\right)$ [14]. By substituting the cdf of EHL distribution given in (1) into equation (11) and differentiating with respect to α and β, the LS estimators of the parameters are obtained by solving the resulting system of nonlinear equations:

$$\frac{\partial S}{\partial \alpha} = \sum_{i=1}^{n} \left(F\left(x_{(i)}\right) - h_i\right) \Delta_1\left(x_{(i)}\right) = 0,$$

$$\frac{\partial S}{\partial \beta} = \sum_{i=1}^{n} \left(F\left(x_{(i)}\right) - h_i\right) \Delta_2\left(x_{(i)}\right) = 0,$$

where $\Delta_1(x)$ and $\Delta_2(x)$ are

$$\Delta_1(x) = \left(\frac{1 - e^{-\beta x}}{1 + e^{-\beta x}}\right)^{\alpha} \log\left(\frac{1 - e^{-\beta x}}{1 + e^{-\beta x}}\right), \quad (12)$$

$$\Delta_2(x) = \alpha \left(\frac{1 - e^{-\beta x}}{1 + e^{-\beta x}}\right)^{\alpha - 1} \frac{2x e^{-\beta x}}{(1 + e^{-\beta x})^2}, \quad (13)$$

respectively.

Weighted Least Squares Estimators

The WLS estimators α and β are derived by minimizing the following expression with respect to the parameters of interest

$$W = \sum_{i=1}^{n} w_i \left(F\left(x_{(i)}\right) - h_i\right)^2. \quad (14)$$

Here, $w_i = 1/V\left(F\left(x_{(i)}\right)\right) = (n+1)^2(n+2)/i(n-i+1), i = 1, \ldots, n$ [15]. By substituting the cdf of the EHL distribution, as defined in equation (1), into equation (14), and taking the partial derivatives with respect to α and β, one obtains a system of nonlinear equations. The WLS estimates of the parameters are then derived by solving these equations:

$$\frac{\partial W}{\partial \alpha} = \sum_{i=1}^{n} w_i^* \left(F\left(x_{(i)}\right) - h_i\right) \Delta_1\left(x_{(i)}\right) = 0,$$

$$\frac{\partial W}{\partial \beta} = \sum_{i=1}^{n} w_i^* \left(F\left(x_{(i)}\right) - h_i\right) \Delta_2\left(x_{(i)}\right) = 0,$$

where $w_i^* = \frac{1}{i(n-i+1)}$, and $\Delta_1(x)$ and $\Delta_2(x)$ are the same as in (12) and (13), respectively.

Cramér-Von Mises Estimators

The CM estimators α and β are derived by minimizing the following objective function with respect to the parameters of interest:

$$C = \frac{1}{12n} + \sum_{i=1}^{n} \left(F\left(x_{(i)}\right) - t_i\right)^2. \tag{15}$$

Here, $t_i = \frac{2i-1}{2n}, i = 1, \dots, n$ [16]. The CM estimators of the parameters are obtained by solving the resulting system of nonlinear equations:

$$\frac{\partial C}{\partial \alpha} = \sum_{i=1}^{n} \left(F\left(x_{(i)}\right) - t_i\right) \Delta_1\left(x_{(i)}\right) = 0,$$

$$\frac{\partial C}{\partial \beta} = \sum_{i=1}^{n} \left(F\left(x_{(i)}\right) - t_i\right) \Delta_2\left(x_{(i)}\right) = 0,$$

where $\Delta_1(x)$ and $\Delta_2(x)$ are the same as in (12) and (13), respectively.

Anderson Darling Estimators

The AD estimators α and β are derived by minimizing the following objective function with respect to the parameters of interest

$$A = -n - \frac{1}{n} \sum_{i=1}^{n} k_i \log\left[F\left(x_{(i)}\right)\left(1 - F\left(x_{(i^*)}\right)\right)\right], \tag{16}$$

where $k_i = (2i-1), i = 1, \dots, n; i^* = n - i + 1$, moreover $x_{(i^*)}$ represents decreasing order of magnitude [17].

The AD estimators of the parameters are derived by solving the following nonlinear equations

$$\frac{\partial A}{\partial \alpha} = \sum_{i=1}^{n} k_i \left[\frac{\Delta_1\left(x_{(i)}\right)}{F\left(x_{(i)}\right)} - \frac{\Delta_1\left(x_{(i^*)}\right)}{1 - F\left(x_{(i^*)}\right)}\right] = 0,$$

$$\frac{\partial A}{\partial \beta} = \sum_{i=1}^{n} k_i \left[\frac{\Delta_2\left(x_{(i)}\right)}{F\left(x_{(i)}\right)} - \frac{\Delta_2\left(x_{(i^*)}\right)}{1 - F\left(x_{(i^*)}\right)}\right] = 0,$$

where $\Delta_1(x)$ and $\Delta_2(x)$ are the same as in (12) and (13), respectively.

Right-Tail Anderson Darling Estimators

The RTAD estimators of α and β are obtained by minimizing the following expression with respect to the parameters of interest [18]:

$$R = \frac{n}{2} - 2\sum_{i=1}^{n} F\left(x_{(i)}\right) - \frac{1}{n}\sum_{i=1}^{n} k_i \log\left(1 - F\left(x_{(i^*)}\right)\right), \tag{17}$$

The RTAD estimators of α and β are obtained by solving the resulting system of nonlinear equations:

$$\frac{\partial R}{\partial \alpha} = -2\sum_{i=1}^{n} \Delta_1\left(x_{(i)}\right) + \frac{1}{n}\sum_{i=1}^{n} k_i \frac{\Delta_1\left(x_{(i^*)}\right)}{1 - F\left(x_{(i^*)}\right)} = 0,$$

$$\frac{\partial R}{\partial \beta} = -2\sum_{i=1}^{n} \Delta_2\left(x_{(i)}\right) + \frac{1}{n}\sum_{i=1}^{n} k_i \frac{\Delta_2\left(x_{(i^*)}\right)}{1 - F\left(x_{(i^*)}\right)} = 0,$$

where k_i is defined as in previous subsection, also $\Delta_1(x)$ and $\Delta_2(x)$ are the same as in (12) and (13), respectively.

Left-Tail Anderson Darling Estimators

The LTAD estimation method involves parameter estimation based on a modification of the AD statistic that emphasizes the fit in the left tail of the distribution [19]. The estimators of α and β under the LTAD approach are obtained by minimizing the following objective function with respect to the parameters:

$$LT = -\frac{3}{2}n + 2\sum_{i=1}^{n} F\left(x_{(i)}\right) - \frac{1}{n}\sum_{i=1}^{n} k_i \log F\left(x_{(i)}\right). \tag{18}$$

The LTAD estimators of α and β are obtained by solving the resulting system of nonlinear equations:

$$\frac{\partial LT}{\partial \alpha} = 2\sum_{i=1}^{n} \Delta_1\left(x_{(i)}\right) - \frac{1}{n}\sum_{i=1}^{n} k_i \frac{\Delta_1\left(x_{(i)}\right)}{F\left(x_{(i)}\right)} = 0,$$

$$\frac{\partial LT}{\partial \beta} = 2\sum_{i=1}^{n} \Delta_2\left(x_{(i)}\right) - \frac{1}{n}\sum_{i=1}^{n} k_i \frac{\Delta_2\left(x_{(i)}\right)}{F\left(x_{(i)}\right)} = 0,$$

where k_i is defined as in previous subsection, also $\Delta_1(x)$ and $\Delta_2(x)$ are the same as in (12) and (13), respectively.

Anderson Darling Left-Tail Second-Order Estimators

The ADLTS estimators α and β are derived by minimizing the following expression with respect to the parameters of interest

$$LTS = 2\sum_{i=1}^{n} \log F(x_{(i)}) + \frac{1}{n}\sum_{i=1}^{n} \frac{k_i}{F(x_{(i)})}. \tag{19}$$

The ADLTS estimators of the parameters are derived by solving the following nonlinear equations

$$\frac{\partial LTS}{\partial \alpha} = 2\sum_{i=1}^{n} \frac{\Delta_1\left(x_{(i)}\right)}{F\left(x_{(i)}\right)} - \frac{1}{n}\sum_{i=1}^{n} \frac{k_i \Delta_1\left(x_{(i)}\right)}{F\left(x_{(i)}\right)^2} = 0,$$

$$\frac{\partial LTS}{\partial \beta} = 2\sum_{i=1}^{n} \frac{\Delta_2\left(x_{(i)}\right)}{F\left(x_{(i)}\right)} - \frac{1}{n}\sum_{i=1}^{n} \frac{k_i \Delta_2\left(x_{(i)}\right)}{F\left(x_{(i)}\right)^2} = 0,$$

where k_i is defined as in previous subsection, also $\Delta_1(x)$ and $\Delta_2(x)$ are the same as in (12) and (13), respectively.

Maximum Product of Spacings Estimators

The MPS method is a widely used nonparametric estimation technique for estimating the parameters of probability distribution. It relies on maximizing the product of the observed spacings between ordered data points, [20]. The MPS estimators of the parameters α and β are obtained by maximizing the following objective function with respect to these parameters

$$\delta = \frac{1}{n+1}\sum_{i=1}^{n+1} \log\varpi_i\left(x_{(i)}\right), \tag{20}$$

where $\varpi_i\left(x_{(i)}\right) = F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right), F\left(x_{(0)}\right) = 0$ and $F\left(x_{(n+1)}\right) = 1$.
The MPS estimators of α and β are derived by solving the resulting system of nonlinear equations

$$\frac{\partial\delta}{\partial\alpha} = \sum_{i=1}^{n+1} \frac{\Delta_1\left(x_{(i)}\right) - \Delta_1\left(x_{(i-1)}\right)}{F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)} = 0,$$

$$\frac{\partial\delta}{\partial\beta} = \sum_{i=1}^{n+1} \frac{\Delta_2\left(x_{(i)}\right) - \Delta_2\left(x_{(i-1)}\right)}{F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)} = 0,$$

where $\Delta_1(x,\alpha,\beta)$ and $\Delta_2(x,\alpha,\beta)$ are the same as in (12) and (13), respectively.

Minimum Spacing Absolute Distance Estimators

The MSAD estimators of the parameters α and β are obtained by minimizing the following expression with respect to these parameters [21]:

$$\zeta = \sum_{i=1}^{n+1} \left|\varpi_i - \frac{1}{n+1}\right|, \tag{21}$$

The MSAD estimators of α and β are obtained by solving the resulting system of nonlinear equations

$$\frac{\partial\zeta}{\partial\alpha} = \sum_{i=1}^{n+1} \frac{F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right) - \frac{1}{n+1}}{\left|F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right) - \frac{1}{n+1}\right|}\left[\Delta_1\left(x_{(i)}\right) - \Delta_1\left(x_{(i-1)}\right)\right] = 0,$$

$$\frac{\partial\zeta}{\partial\beta} = \sum_{i=1}^{n+1} \frac{F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right) - \frac{1}{n+1}}{\left|F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right) - \frac{1}{n+1}\right|}\left[\Delta_2\left(x_{(i)}\right) - \Delta_2\left(x_{(i-1)}\right)\right] = 0,$$

where $\Delta_1(x,\alpha,\beta)$ and $\Delta_2(x,\alpha,\beta)$ are the same as in (12) and (13), respectively.

Minimum Spacing Absolute-Log Distance Estimators

The MSALD estimators of the parameters α and β are obtained by minimizing the following equation with respect to the parameters of interest [22]:

$$\gamma = \sum_{i=1}^{n+1} \left|\log\varpi_i - \log\frac{1}{n+1}\right|, \tag{22}$$

The MSALD estimators of the parameters are derived by solving the following nonlinear equations

$$\frac{\partial\gamma}{\partial\alpha} = \sum_{i=1}^{n+1} \frac{\log\left(F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)\right) - \log\left(\frac{1}{n+1}\right)}{\left|\log\left(F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)\right) - \log\left(\frac{1}{n+1}\right)\right|}\left[\frac{\Delta_1\left(x_{(i)}\right) - \Delta_1\left(x_{(i-1)}\right)}{F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)}\right] = 0,$$

$$\frac{\partial\gamma}{\partial\beta} = \sum_{i=1}^{n+1} \frac{\log\left(F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)\right) - \log\left(\frac{1}{n+1}\right)}{\left|\log\left(F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)\right) - \log\left(\frac{1}{n+1}\right)\right|}\left[\frac{\Delta_2\left(x_{(i)}\right) - \Delta_2\left(x_{(i-1)}\right)}{F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)}\right] = 0,$$

where $\Delta_1(x)$ and $\Delta_2(x)$ are the same as in (12) and (13), respectively.

Simulation Study

A Monte Carlo simulation is conducted to compare the performance of proposed methods examined in this study. For each parameter, the bias and root mean square error (RMSE) values of the estimators are calculated. The RMSEs of $\hat{\alpha}$ and $\hat{\beta}$ are calculated from the following equalities, respectively

$$RMSE(\hat{\alpha}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2} \text{ and } RMSE(\hat{\beta}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\beta}_i - \beta)^2}. \quad (23)$$

In the simulation study, the inverse transformation method is used to generate random numbers from the EHL distribution. It is defined as shown below

$$x = {F_x}^{-1}(u) = -\frac{1}{\beta}\log\left(\frac{1 - u^{\frac{1}{\alpha}}}{1 + u^{\frac{1}{\alpha}}}\right), \qquad U \sim Uniform(0,1) \quad (24)$$

The performances of the estimators are compared under different scenarios when the sample sizes are 25, 50, 100, and 200, the shape parameter values α are 0.5, 1, and 3, and the scale parameter values β are 1 and 3. The number of simulation replications is taken as 10.000. All analyses are carried out in MATLAB R2024b using the *fminsearch* function, which implements the Nelder–Mead simplex algorithm. For the ML method, fixed initial values are employed. Moreover, these estimates are utilized as the initial value to obtain the remaining methods. Default convergence tolerances of the optimization routine are adopted, with a maximum of 5,000 iterations. During simulations, convergence diagnostics are monitored, and non-convergent runs are excluded from subsequent analyses. The proportion of such cases is negligible and do not affect the overall conclusions. The simulated bias and RMSE values of different estimators for the parameters of the EHL distribution are presented in Tables 1–6. According to these tables, in view of bias values of the estimators or in other words the biases of the estimators, the LS and the WLS estimators performs better than the remaining estimators, especially for small sample sizes. They are followed by the MSAD and the AD estimators. On the other hand, the CM and the RTAD estimators exhibit the worst performance with the highest bias values of the estimates. Moreover, the ML and the MPS estimator have larger biases than the other methods. However, as the sample sizes increase, all estimators have negligible bias, as expected. Regarding the RMSE values in Tables 1-6, the MPS estimator outperforms other estimators for all sample sizes. Moreover, the ML and the AD estimators perform better than the remaining estimators in most cases. By contrast, the ADLTS, the CM, the MSAD and the RTAD estimators demonstrate the weakest performance with the highest RMSE values especially for small sample sizes. As the sample sizes increase, the RMSE values of all estimators decrease as expected. Table 7 summarizes the RMSE rankings of the estimation methods under different parameter settings and sample sizes. According to the overall ranking results, the MPS estimator exhibits the most efficient performance across all parameter settings and sample sizes, followed by the ML and AD estimators. In contrast, the ADLTS method shows the weakest overall performance. These findings are expected to provide guidance for selecting the most appropriate estimation method in future applications and academic studies.

Table 1. *Simulated bias and RMSE values of $\hat{\alpha}$ and $\hat{\beta}$; $\alpha = 0.5,\ \beta = 1$*

n	Est.	Est.Par.	ML	LS	WLS	CM	AD	RTAD	LTAD	ADLTS	MPS	MSAD	MSALD
25	BIAS	$\hat{\alpha}$	0.0435	0.0104	0.0147	0.0515	0.0213	0.0422	0.0316	0.0296	-0.0299	0.0148	-0.0229
		$\hat{\beta}$	0.1056	0.0144	0.0287	0.1228	0.0495	0.0670	0.1107	0.1915	-0.0783	0.0220	-0.0617
	RMSE	$\hat{\alpha}$	0.1470[4]	0.1555[7]	0.1472[5]	0.1829[10]	0.1407[3]	0.1825[9]	0.1553[6]	0.1909[11]	0.1203[1]	0.1791[8]	0.1359[2]
		$\hat{\beta}$	0.3377[4]	0.3680[7]	0.3468[6]	0.4247[9]	0.3299[3]	0.3388[5]	0.4623[10]	0.8281[11]	0.2866[1]	0.3731[8]	0.3191[2]
	$\sum Ranks$		8[4]	14[6.5]	11[5]	19[10]	6[3]	14[6.5]	16[8.5]	22[11]	2[1]	16[8.5]	4[2]
50	BIAS	$\hat{\alpha}$	0.0204	0.0046	0.0086	0.0236	0.0104	0.0203	0.0143	0.0019	-0.0222	-0.0022	-0.0183
		$\hat{\beta}$	0.0483	0.0040	0.0160	0.0554	0.0221	0.0309	0.0485	0.0466	-0.0585	-0.0078	-0.0498
	RMSE	$\hat{\alpha}$	0.0908[2]	0.0995[7]	0.0937[5]	0.1076[8]	0.0916[3]	0.1128[9]	0.0960[6]	0.1150[11]	0.0825[1]	0.1131[10]	0.0927[4]
		$\hat{\beta}$	0.2061[2]	0.2334[7]	0.2160[6]	0.2512[8]	0.2092[3]	0.2140[4]	0.2760[10]	0.4636[11]	0.1928[1]	0.2551[9]	0.2158[5]
	$\sum Ranks$		4[2]	14[7]	11[5]	16[8.5]	6[3]	13[6]	16[8.5]	22[11]	2[1]	19[10]	9[4]
100	BIAS	$\hat{\alpha}$	0.0101	0.0022	0.0050	0.0114	0.0052	0.0096	0.0073	-0.0060	-0.0145	-0.0055	-0.0119
		$\hat{\beta}$	0.0230	0.0014	0.0094	0.0265	0.0101	0.0141	0.0238	-0.0063	-0.0385	-0.0141	-0.0324
	RMSE	$\hat{\alpha}$	0.0612[2]	0.0677[7]	0.0634[4]	0.0704[8]	0.0626[3]	0.0738[9]	0.0653[5]	0.0818[11]	0.0587[1]	0.0767[10]	0.0672[6]
		$\hat{\beta}$	0.1395[2]	0.1626[7]	0.1487[5]	0.1686[8]	0.1458[3]	0.1472[4]	0.1859[10]	0.3109[11]	0.1364[1]	0.1778[9]	0.1554[6]
	$\sum Ranks$		4[2]	14[7]	9[4]	16[9]	6[3]	13[6]	15[8]	22[11]	2[1]	19[10]	12[5]
200	BIAS	$\hat{\alpha}$	0.0058	0.0014	0.0033	0.0059	0.0031	0.0048	0.0044	-0.0062	-0.0082	-0.0038	-0.0070
		$\hat{\beta}$	0.0132	0.0020	0.0071	0.0144	0.0068	0.0081	0.0143	-0.0160	-0.0219	-0.0114	-0.0190
	RMSE	$\hat{\alpha}$	0.0415[2]	0.0468[6]	0.0433[4]	0.0478[8]	0.0430[3]	0.0497[9]	0.0447[5]	0.0595[11]	0.0406[1]	0.0525[10]	0.0472[7]
		$\hat{\beta}$	0.0958[2]	0.1116[7]	0.1014[5]	0.1138[8]	0.1002[3]	0.1010[4]	0.1261[10]	0.2265[11]	0.0948[1]	0.1220[9]	0.1096[6]
	$\sum Ranks$		4[2]	13[6]	9[4]	16[9]	6[3]	13[6]	15[8]	22[11]	2[1]	19[10]	13[6]

Table 2. *Simulated bias and RMSE values of $\hat{\alpha}$ and $\hat{\beta}$; $\alpha = 1,\ \beta = 1$*

n	Est.	Est.Par.	ML	LS	WLS	CM	AD	RTAD	LTAD	ADLTS	MPS	MSAD	MSALD
25	BIAS	$\hat{\alpha}$	0.1178	0.0354	0.0455	0.1451	0.0614	0.1267	0.0895	0.1032	-0.0644	-0.0041	-0.0449
		$\hat{\beta}$	0.0712	-0.0029	0.0094	0.0780	0.0268	0.0451	0.0603	0.0911	-0.0736	-0.0269	-0.0596
	RMSE	$\hat{\alpha}$	0.3510[4]	0.3851[8]	0.3605[6]	0.4809[9]	0.3286[3]	0.5034[11]	0.3722[7]	0.4887[10]	0.2666[1]	0.3536[5]	0.3026[2]
		$\hat{\beta}$	0.2467[4]	0.2686[8]	0.2537[5]	0.3004[9]	0.2396[2]	0.2651[6]	0.3013[10]	0.4662[11]	0.2225[1]	0.2651[7]	0.2454[3]
	$\sum Ranks$		8[4]	16[7]	11[5]	18[10]	5[2.5]	17[8.5]	17[8.5]	21[11]	2[1]	12[6]	5[2.5]
50	BIAS	$\hat{\alpha}$	0.0522	0.0127	0.0222	0.0615	0.0273	0.0542	0.0380	0.0177	-0.0512	-0.0177	-0.0406
		$\hat{\beta}$	0.0335	-0.0034	0.0071	0.0357	0.0124	0.0214	0.0270	0.0161	-0.0513	-0.0246	-0.0427
	RMSE	$\hat{\alpha}$	0.2109[3]	0.2338[7]	0.2179[5]	0.2586[9]	0.2110[4]	0.2694[10]	0.2242[6]	0.2698[11]	0.1852[1]	0.2409[8]	0.2097[2]
		$\hat{\beta}$	0.1603[2]	0.1772[7]	0.1655[4]	0.1873[8]	0.1612[3]	0.1730[5]	0.1942[10]	0.2935[11]	0.1552[1]	0.1907[9]	0.1747[6]
	$\sum Ranks$		5[2]	14[6]	9[5]	17[9.5]	7[3]	15[7]	16[8]	22[11]	2[1]	17[9.5]	8[4]
100	BIAS	$\hat{\alpha}$	0.0230	0.0032	0.0099	0.0265	0.0110	0.0223	0.0164	-0.0104	-0.0363	-0.0173	-0.0295
		$\hat{\beta}$	0.0164	-0.0016	0.0051	0.0177	0.0062	0.0100	0.0136	-0.0125	-0.0326	-0.0188	-0.0280
	RMSE	$\hat{\alpha}$	0.1370[2]	0.1540[7]	0.1428[4]	0.1616[8]	0.1405[3]	0.1704[10]	0.1469[5]	0.1827[11]	0.1301[1]	0.1663[9]	0.1492[6]
		$\hat{\beta}$	0.1104[2]	0.1252[7]	0.1157[4]	0.1287[8]	0.1139[3]	0.1205[5]	0.1330[9]	0.2073[11]	0.1098[1]	0.1372[10]	0.1252[6]
	$\sum Ranks$		4[2]	14[6.5]	8[4]	16[9]	6[3]	15[8]	14[6.5]	22[11]	2[1]	19[10]	12[5]
200	BIAS	$\hat{\alpha}$	0.0141	0.0040	0.0082	0.0155	0.0078	0.0128	0.0108	-0.0134	-0.0197	-0.0106	-0.0168
		$\hat{\beta}$	0.0091	-0.0001	0.0041	0.0095	0.0037	0.0053	0.0078	-0.0188	-0.0188	-0.0125	-0.0168
	RMSE	$\hat{\alpha}$	0.0943[2]	0.1081[7]	0.0990[4]	0.1110[8]	0.0979[3]	0.1162[9]	0.1021[5]	0.1325[11]	0.0914[1]	0.1162[10]	0.1052[6]
		$\hat{\beta}$	0.0769[1]	0.0884[7]	0.0807[4]	0.0897[8]	0.0799[3]	0.0840[5]	0.0931[9]	0.1551[11]	0.0770[2]	0.0963[10]	0.0880[6]
	$\sum Ranks$		3[1.5]	14[7]	8[4]	16[9]	6[3]	14[7]	14[7]	22[11]	3[1.5]	20[10]	12[5]

Table 3. *Simulated bias and RMSE values of $\hat{\alpha}$ and $\hat{\beta}$; $\alpha = 3,\ \beta = 1$*

n	Est.	Est.Par.	ML	LS	WLS	CM	AD	RTAD	LTAD	ADLTS	MPS	MSAD	MSALD
25	BIAS	$\hat{\alpha}$	0.6122	0.2684	0.2973	0.8461	0.3283	0.7989	0.4764	0.8171	-0.2308	-0.1331	-0.1250
		$\hat{\beta}$	0.0577	-0.0084	0.0029	0.0585	0.0164	0.0380	0.0372	0.0619	-0.0651	-0.0666	-0.0541
	RMSE	$\hat{\alpha}$	1.7015[4]	2.0613[8]	1.9113[6]	2.8955[9]	1.5568[3]	3.3160[10]	1.9382[7]	4.2980[11]	1.0949[1]	1.8403[5]	1.3123[2]
		$\hat{\beta}$	0.1975[3]	0.2145[6]	0.2022[5]	0.2381[10]	0.1899[2]	0.2291[8]	0.2193[7]	0.3148[11]	0.1814[1]	0.2350[9]	0.2012[4]
	$\sum Ranks$		7[4]	14[7]	11[5]	19[10]	5[2]	18[9]	14[7]	22[11]	2[1]	14[7]	6[3]
50	BIAS	$\hat{\alpha}$	0.2793	0.1199	0.1502	0.3559	0.1612	0.3248	0.2119	0.2069	-0.1844	-0.0780	-0.1285
		$\hat{\beta}$	0.0287	-0.0023	0.0062	0.0302	0.0095	0.0198	0.0182	0.0126	-0.0437	-0.0339	-0.0382
	RMSE	$\hat{\alpha}$	0.9385[4]	1.0747[8]	0.9785[5]	1.2512[9]	0.9179[3]	1.3231[11]	0.9881[6]	1.2574[10]	0.7467[1]	1.0705[7]	0.8555[2]
		$\hat{\beta}$	0.1350[2]	0.1510[7]	0.1403[4]	0.1593[9]	0.1358[3]	0.1553[8]	0.1507[6]	0.2060[11]	0.1312[1]	0.1697[10]	0.1468[5]
	$\sum Ranks$		6[2.5]	15[7]	9[5]	18[9]	6[2.5]	19[10]	12[6]	21[11]	2[1]	17[8]	7[4]
100	BIAS	$\hat{\alpha}$	0.1275	0.0519	0.0742	0.1604	0.0726	0.1378	0.1012	0.0423	-0.1345	-0.0601	-0.1026
		$\hat{\beta}$	0.0143	-0.0006	0.0049	0.0155	0.0050	0.0090	0.0103	-0.0036	-0.0279	-0.0187	-0.0245
	RMSE	$\hat{\alpha}$	0.5530[2]	0.6642[7]	0.5943[5]	0.7163[9]	0.5730[4]	0.7619[11]	0.6052[6]	0.7553[10]	0.4959[1]	0.6690[8]	0.5672[3]
		$\hat{\beta}$	0.0891[1]	0.1031[7]	0.0942[4]	0.1061[9]	0.0922[3]	0.1052[8]	0.1015[5]	0.1452[11]	0.0891[2]	0.1143[10]	0.1015[6]
	$\sum Ranks$		3[1.5]	14[7]	9[4.5]	18[8.5]	7[3]	19[10]	11[6]	21[11]	3[1.5]	18[8.5]	9[4.5]
200	BIAS	$\hat{\alpha}$	0.0620	0.0255	0.0411	0.0779	0.0371	0.0726	0.0475	-0.0337	-0.0874	-0.0393	-0.0745
		$\hat{\beta}$	0.0074	-0.0001	0.0034	0.0079	0.0028	0.0054	0.0049	-0.0138	-0.0169	-0.0110	-0.0155
	RMSE	$\hat{\alpha}$	0.3678[2]	0.4459[7]	0.3973[4]	0.4633[9]	0.3887[3]	0.5049[10]	0.4003[5]	0.5129[11]	0.3497[1]	0.4589[8]	0.4018[6]
		$\hat{\beta}$	0.0618[1]	0.0720[7]	0.0654[4]	0.0730[8]	0.0645[3]	0.0735[9]	0.0704[5]	0.1081[11]	0.0623[2]	0.0794[10]	0.0715[6]
	$\sum Ranks$		3[1.5]	14[7]	8[4]	17[8]	6[3]	19[10]	10[5]	22[11]	3[1.5]	18[9]	12[6]

Table 4. *Simulated bias and RMSE values of $\hat{\alpha}$ and $\hat{\beta}$; $\alpha = 0.5,\ \beta = 3$*

n	Est.	Est.Par.	ML	LS	WLS	CM	AD	RTAD	LTAD	ADLTS	MPS	MSAD	MSALD
25	BIAS	$\hat{\alpha}$	0.0422	0.0074	0.0121	0.0483	0.0191	0.0395	0.0305	0.0303	-0.0309	-0.0304	-0.0254
		$\hat{\beta}$	0.3029	0.0271	0.0687	0.3512	0.1319	0.1810	0.3318	0.5815	-0.2459	-0.2757	-0.1932
	RMSE	$\hat{\alpha}$	0.1446[5]	0.1523[7]	0.1441[4]	0.1795[9]	0.1376[3]	0.1836[10]	0.1516[6]	0.1955[11]	0.1191[1]	0.1543[8]	0.1334[2]
		$\hat{\beta}$	0.9990[4]	1.0885[8]	1.0201[6]	1.2526[9]	0.9680[3]	1.0090[5]	1.3827[10]	2.6078[11]	0.8558[1]	1.0359[7]	0.9457[2]
	$\sum Ranks$		9[4]	15[7]	10[5]	18[10]	6[3]	15[7]	16[9]	22[11]	2[1]	15[7]	4[2]
50	BIAS	$\hat{\alpha}$	0.0223	0.0068	0.0106	0.0259	0.0122	0.0213	0.0172	0.0053	-0.0205	-0.0134	-0.0159
		$\hat{\beta}$	0.1556	0.0311	0.0644	0.1863	0.0798	0.1022	0.1709	0.1632	-0.1656	-0.1384	-0.1359
	RMSE	$\hat{\alpha}$	0.0911[3]	0.0992[7]	0.0936[4]	0.1078[8]	0.0911[2]	0.1110[9]	0.0976[6]	0.1195[11]	0.0821[1]	0.1116[10]	0.0938[5]
		$\hat{\beta}$	0.6371[2]	0.7313[7]	0.6818[6]	0.7892[8]	0.6520[3]	0.6577[4]	0.8719[10]	1.4186[11]	0.5897[1]	0.7998[9]	0.6606[5]
	$\sum Ranks$		5[2.5]	14[7]	10[4.5]	16[8.5]	5[2.5]	13[6]	16[8.5]	22[11]	2[1]	19[10]	10[4.5]
100	BIAS	$\hat{\alpha}$	0.0100	0.0025	0.0052	0.0117	0.0053	0.0100	0.0071	-0.0076	-0.0146	-0.0084	-0.0123
		$\hat{\beta}$	0.0696	0.0037	0.0278	0.0791	0.0300	0.0433	0.0670	-0.0344	-0.1147	-0.0664	-0.0941
	RMSE	$\hat{\alpha}$	0.0607[2]	0.0682[7]	0.0637[4]	0.0710[8]	0.0626[3]	0.0735[9]	0.0655[5]	0.0808[11]	0.0582[1]	0.0764[10]	0.0666[6]
		$\hat{\beta}$	0.4126[2]	0.4864[7]	0.4436[5]	0.5045[8]	0.4336[4]	0.4321[3]	0.5624[10]	0.9223[11]	0.4033[1]	0.5266[9]	0.4599[6]
	$\sum Ranks$		4[2]	14[7]	9[4]	16[9]	7[3]	12[5.5]	15[8]	22[11]	2[1]	19[10]	12[5.5]
200	BIAS	$\hat{\alpha}$	0.0041	0.0001	0.0019	0.0047	0.0016	0.0040	0.0025	-0.0092	-0.0099	-0.0058	-0.0085
		$\hat{\beta}$	0.0351	0.0010	0.0164	0.0383	0.0150	0.0218	0.0330	-0.0705	-0.0700	-0.0418	-0.0610
	RMSE	$\hat{\alpha}$	0.0413[2]	0.0469[6]	0.0434[4]	0.0477[8]	0.0431[3]	0.0500[9]	0.0447[5]	0.0593[11]	0.0409[1]	0.0525[10]	0.0470[7]
		$\hat{\beta}$	0.2878[2]	0.3392[7]	0.3069[5]	0.3453[8]	0.3034[3]	0.3036[4]	0.3827[10]	0.6801[11]	0.2865[1]	0.3640[9]	0.3311[6]
	$\sum Ranks$		4[2]	13[6]	9[4]	16[9]	6[3]	13[6]	15[8]	22[11]	2[1]	19[10]	13[6]

Table 5. *Simulated bias and RMSE values of $\hat{\alpha}$ and $\hat{\beta}$; $\alpha = 1,\ \beta = 3$*

n	Est.	Est.Par.	ML	LS	WLS	CM	AD	RTAD	LTAD	ADLTS	MPS	MSAD	MSALD
25	BIAS	$\hat{\alpha}$	0.1143	0.0303	0.0416	0.1386	0.0584	0.1236	0.0845	0.0958	-0.0670	-0.0512	-0.0485
		$\hat{\beta}$	0.2165	-0.0091	0.0314	0.2330	0.0838	0.1375	0.1839	0.2704	-0.2187	-0.2171	-0.1780
	RMSE	$\hat{\alpha}$	0.3500[4]	0.3768[7]	0.3514[5]	0.4681[9]	0.3273[3]	0.4849[11]	0.3606[6]	0.4738[10]	0.2676[1]	0.3876[8]	0.3045[2]
		$\hat{\beta}$	0.7422[3]	0.7940[7]	0.7554[5]	0.8884[8]	0.7180[2]	0.7881[6]	0.9048[10]	1.3969[11]	0.6684[1]	0.8913[9]	0.7423[4]
	$\sum Ranks$		7[4]	14[6]	10[5]	17[9]	5[2]	17[9]	16[7]	21[11]	2[1]	17[9]	6[3]
50	BIAS	$\hat{\alpha}$	0.0557	0.0169	0.0265	0.0661	0.0313	0.0581	0.0417	0.0207	-0.0482	-0.0220	-0.0378
		$\hat{\beta}$	0.1088	0.0006	0.0321	0.1179	0.0470	0.0726	0.0908	0.0534	-0.1461	-0.0916	-0.1211
	RMSE	$\hat{\alpha}$	0.2134[4]	0.2357[7]	0.2199[5]	0.2615[9]	0.2128[3]	0.2745[11]	0.2246[6]	0.2716[10]	0.1861[1]	0.2530[8]	0.2085[2]
		$\hat{\beta}$	0.4892[2]	0.5407[7]	0.5064[4]	0.5730[8]	0.4928[3]	0.5279[6]	0.5889[9]	0.8780[11]	0.4690[1]	0.6115[10]	0.5248[5]
	$\sum Ranks$		6[2.5]	14[6]	9[5]	17[8.5]	6[2.5]	17[8.5]	15[7]	21[11]	2[1]	18[10]	7[4]
100	BIAS	$\hat{\alpha}$	0.0254	0.0073	0.0135	0.0308	0.0139	0.0278	0.0181	-0.0079	-0.0339	-0.0156	-0.0280
		$\hat{\beta}$	0.0541	0.0014	0.0215	0.0593	0.0235	0.0395	0.0413	-0.0341	-0.0930	-0.0557	-0.0818
	RMSE	$\hat{\alpha}$	0.1357[2]	0.1542[7]	0.1418[4]	0.1625[8]	0.1386[3]	0.1719[10]	0.1452[5]	0.1841[11]	0.1281[1]	0.1696[9]	0.1468[6]
		$\hat{\beta}$	0.3279[2]	0.3719[7]	0.3424[4]	0.3834[8]	0.3359[3]	0.3618[5]	0.3930[9]	0.6241[11]	0.3249[1]	0.4208[10]	0.3716[6]
	$\sum Ranks$		4[2]	14[6.5]	8[4]	16[9]	6[3]	15[8]	14[6.5]	22[11]	2[1]	19[10]	12[5]
200	BIAS	$\hat{\alpha}$	0.0127	0.0035	0.0075	0.0149	0.0069	0.0125	0.0098	-0.0138	-0.0211	-0.0099	-0.0175
		$\hat{\beta}$	0.0247	-0.0010	0.0109	0.0277	0.0097	0.0153	0.0218	-0.0543	-0.0589	-0.0348	-0.0521
	RMSE	$\hat{\alpha}$	0.0936[2]	0.1080[7]	0.0990[4]	0.1108[8]	0.0978[3]	0.1174[9]	0.1021[5]	0.1317[11]	0.0911[1]	0.1174[10]	0.1044[6]
		$\hat{\beta}$	0.2287[1]	0.2612[6]	0.2399[4]	0.2650[7]	0.2374[3]	0.2506[5]	0.2791[9]	0.4672[11]	0.2299[2]	0.2953[10]	0.2674[8]
	$\sum Ranks$		3[1.5]	13[5]	8[4]	15[9]	6[3]	14[7]	14[7]	22[11]	3[1.5]	20[10]	14[7]

Table 6. *Simulated bias and RMSE values of $\hat{\alpha}$ and $\hat{\beta}$; $\alpha = 3,\ \beta = 3$*

n	Est.	Est.Par.	ML	LS	WLS	CM	AD	RTAD	LTAD	ADLTS	MPS	MSAD	MSALD
25	BIAS	$\hat{\alpha}$	0.5939	0.2856	0.3046	0.8686	0.3286	0.7732	0.4842	0.7136	-0.2462	-0.0894	-0.1555
		$\hat{\beta}$	0.1731	-0.0116	0.0189	0.1905	0.0565	0.1202	0.1183	0.1629	-0.1965	-0.1701	-0.1711
	RMSE	$\hat{\alpha}$	1.6452[4]	2.0967[8]	1.8698[6]	2.9420[10]	1.5006[3]	2.8659[9]	1.8314[5]	3.8473[11]	1.0759[1]	2.0244[7]	1.2449[2]
		$\hat{\beta}$	0.5979[3]	0.6589[6]	0.6213[5]	0.7353[10]	0.5799[2]	0.6958[8]	0.6700[7]	0.9108[11]	0.5497[1]	0.7109[9]	0.6060[4]
	$\sum Ranks$		7[4]	14[7]	11[5]	20[10]	5[2]	17[9]	12[6]	22[11]	2[1]	16[8]	6[3]
50	BIAS	$\hat{\alpha}$	0.2652	0.1163	0.1445	0.3516	0.1547	0.3283	0.2004	0.1794	-0.1957	-0.0679	-0.1460
		$\hat{\beta}$	0.0886	-0.0018	0.0233	0.0960	0.0328	0.0670	0.0569	0.0330	-0.1289	-0.0847	-0.1121
	RMSE	$\hat{\alpha}$	0.8848[4]	1.0421[8]	0.9428[5]	1.2132[10]	0.8804[3]	1.2875[11]	0.9472[6]	1.1919[9]	0.7104[1]	1.0317[7]	0.8112[2]
		$\hat{\beta}$	0.3912[2]	0.4430[7]	0.4098[4]	0.4688[9]	0.3962[3]	0.4615[8]	0.4377[6]	0.6018[11]	0.3796[1]	0.4898[10]	0.4291[5]
	$\sum Ranks$		6[2.5]	15[7]	9[5]	19[9.5]	6[2.5]	19[9.5]	12[6]	20[11]	2[1]	17[8]	7[4]
100	BIAS	$\hat{\alpha}$	0.1253	0.0494	0.0732	0.1577	0.0725	0.1426	0.0974	0.0285	-0.1369	-0.0403	-0.1024
		$\hat{\beta}$	0.0443	-0.0010	0.0160	0.0472	0.0166	0.0312	0.0303	-0.0182	-0.0824	-0.0436	-0.0700
	RMSE	$\hat{\alpha}$	0.5503[2]	0.6548[7]	0.5908[5]	0.7061[9]	0.5697[4]	0.7530[11]	0.6005[6]	0.7441[10]	0.4946[1]	0.6725[8]	0.5609[3]
		$\hat{\beta}$	0.2680[2]	0.3083[7]	0.2833[4]	0.3172[9]	0.2772[3]	0.3154[8]	0.3057[6]	0.4367[11]	0.2675[1]	0.3392[10]	0.3016[5]
	$\sum Ranks$		4[2]	14[7]	9[5]	18[8.5]	7[3]	19[10]	12[6]	21[11]	2[1]	18[8.5]	8[4]
200	BIAS	$\hat{\alpha}$	0.0635	0.0287	0.0431	0.0812	0.0392	0.0733	0.0500	-0.0361	-0.0860	-0.0386	-0.0706
		$\hat{\beta}$	0.0216	-0.0002	0.0098	0.0237	0.0081	0.0158	0.0142	-0.0441	-0.0511	-0.0334	-0.0462
	RMSE	$\hat{\alpha}$	0.3668[2]	0.4476[7]	0.3979[4]	0.4653[9]	0.3898[3]	0.4962[10]	0.4054[6]	0.5066[11]	0.3480[1]	0.4568[8]	0.3985[5]
		$\hat{\beta}$	0.1847[1]	0.2167[7]	0.1967[4]	0.2198[9]	0.1942[3]	0.2190[8]	0.2125[5]	0.3208[11]	0.1864[2]	0.2368[10]	0.2127[6]
	$\sum Ranks$		3[1.5]	14[7]	8[4]	18[9]	6[3]	18[9]	11[5.5]	22[11]	3[1.5]	18[9]	11[5.5]

Table 7. *The RMSE rankings of estimation methods under various parameter settings*

Parameter	n	ML	LS	WLS	CM	AD	RTAD	LTAD	ADLTS	MPS	MSAD	MSALD
$\alpha = 0.5, \beta = 1$	25	4.0	6.5	5.0	10.0	3.0	6.5	8.5	11.0	1.0	8.5	2.0
	50	2.0	7.0	5.0	8.5	3.0	6.0	8.5	11.0	1.0	10.0	4.0
	100	2.0	7.0	4.0	9.0	3.0	6.0	8.0	11.0	1.0	10.0	5.0
	200	2.0	6.0	4.0	9.0	3.0	6.0	8.0	11.0	1.0	10.0	6.0
$\alpha = 1, \beta = 1$	25	4.0	7.0	5.0	10.0	2.5	8.5	8.5	11.0	1.0	6.0	2.5
	50	2.0	6.0	5.0	9.5	3.0	7.0	8.0	11.0	1.0	9.5	4.0
	100	2.0	6.5	4.0	9.0	3.0	8.0	6.5	11.0	1.0	10.0	5.0
	200	1.5	7.0	4.0	9.0	3.0	7.0	7.0	11.0	1.5	10.0	5.0
$\alpha = 3, \beta = 1$	25	4.0	7.0	5.0	10.0	2.0	9.0	7.0	11.0	1.0	7.0	3.0
	50	2.5	7.0	5.0	9.0	2.5	10.0	6.0	11.0	1.0	8.0	4.0
	100	1.5	7.0	4.5	8.5	3.0	10.0	6.0	11.0	1.5	8.5	4.5
	200	1.5	7.0	4.0	8.0	3.0	10.0	5.0	11.0	1.5	9.0	6.0
$\alpha = 0.5, \beta = 3$	25	4.0	7.0	5.0	10.0	3.0	7.0	9.0	11.0	1.0	7.0	2.0
	50	2.5	7.0	4.5	8.5	2.5	6.0	8.5	11.0	1.0	10.0	4.5
	100	2.0	7.0	4.0	9.0	3.0	5.5	8.0	11.0	1.0	10.0	5.5
	200	2.0	6.0	4.0	9.0	3.0	6.0	8.0	11.0	1.0	10.0	6.0
$\alpha = 1, \beta = 3$	25	4.0	6.0	5.0	9.0	2.0	9.0	7.0	11.0	1.0	9.0	3.0
	50	2.5	6.0	5.0	8.5	2.5	8.5	7.0	11.0	1.0	10.0	4.0
	100	2.0	6.5	4.0	9.0	3.0	8.0	6.5	11.0	1.0	10.0	5.0
	200	1.5	5.0	4.0	9.0	3.0	7.0	7.0	11.0	1.5	10.0	7.0
$\alpha = 3, \beta = 3$	25	4.0	7.0	5.0	10.0	2.0	9.0	6.0	11.0	1.0	8.0	3.0
	50	2.5	7.0	5.0	9.5	2.5	9.5	6.0	11.0	1.0	8.0	4.0
	100	2.0	7.0	5.0	8.5	3.0	10.0	6.0	11.0	1.0	8.5	4.0
	200	1.5	7.0	4.0	9.0	3.0	9.0	5.5	11.0	1.5	9.0	5.5
$\sum Ranks$		59.5	159.5	109	218.5	66.5	188.5	171.5	264	26.5	216	104.5
Overall Rank		2	6	5	10	3	8	7	11	1	9	4

Real Data Application

Within the scope of the real data application, we model the wind speed data with the EHL distribution. This data set is taken from the NASA Power data access [23]. The wind speed observations are recorded at a height of 50 m daily in the Samandağ region, Türkiye (latitude: 36.08, longitude: 35.98) throughout the year 2019. Before starting to analysis, the suitability of the EHL distribution for the wind speed data is examined by comparing it with commonly used distributions, including Weibull, inverse Weibull (IW), and Burr III. Model comparison is primarily based on the Kolmogorov–Smirnov (KS) statistic, which is used as a relative ranking measure rather than a formal goodness-of-fit test, since model parameters are estimated from the data. Smaller KS values indicate better agreement between the empirical and fitted distributions. In addition, cdf-based RMSE and the coefficient of determination (R^2) are reported as descriptive measures to support the comparative evaluation. Information criteria (Akaike information criterion, AIC and Bayesian information criterion, BIC), computed from the log-likelihood evaluated at the estimated parameters, are mainly emphasized for ML estimation, while results for non-likelihood-based methods are included for comparison [24]. KS, RMSE, R^2, AIC and BIC goodness of fit tests are obtained from the following equations, respectively. $\hat{F}_i$ is the estimated cdf for the *i*th ordered observation. $\bar{\hat{F}} = (1/n)\sum_{i=1}^{n}\hat{F}_i$, n is the sample size, and p is the number of estimated parameters.

$$KS = \max_{1\le i\le n}\left|\frac{i}{n+1} - \hat{F}_i\right|, \tag{25}$$

$$RMSE = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\hat{F}_i - \frac{i}{n+1}\right)^2\right]^{1/2}, \tag{26}$$

$$R^2 = \frac{\sum_{i=1}^{n}\left(\hat{F}_i - \bar{\hat{F}}\right)^2}{\sum_{i=1}^{n}\left(\hat{F}_i - \bar{\hat{F}}\right)^2 + \sum_{i=1}^{n}\left(\hat{F}_i - \frac{i}{n+1}\right)^2}, \tag{27}$$

$$AIC = -2\ln L + 2p, \tag{28}$$

$$BIC = -2\ln L + p\ln n. \tag{29}$$

In Table 8, we give the ML estimators of the unknown parameters of these corresponding distributions and together with model evaluation criteria.

Table 8. *The ML estimates of the unknown parameters and model selection criteria*

	$\hat{\alpha}$	$\hat{\beta}$	KS	RMSE	R^2	AIC	BIC
EHL	7.6345	0.6911	0.0284	0.0117	0.9984	1410.7802	1418.5800
Weibull	2.9217	5.3432	0.0573	0.0290	0.9893	1430.3022	1438.1020
IW	2.5172	3.6765	0.0949	0.0539	0.9577	1502.5827	1510.3825
Burr III	30.1539	2.5921	0.0902	0.0509	0.9629	1492.8878	1500.6876

According to Table 8, the EHL distribution is observed to be the most suitable distribution for modeling wind speed data set with the smallest KS, RMSE, AIC and BIC as well as the highest R^2 values. This distribution is followed by the Weibull, Burr III, and IW distributions, respectively. Moreover, for illustrative purpose, a Q-Q plot is drawn to fit the EHL distribution to the data set. It is observed from Figure 3 that EHL distribution provides good fit to model the wind speed data set.

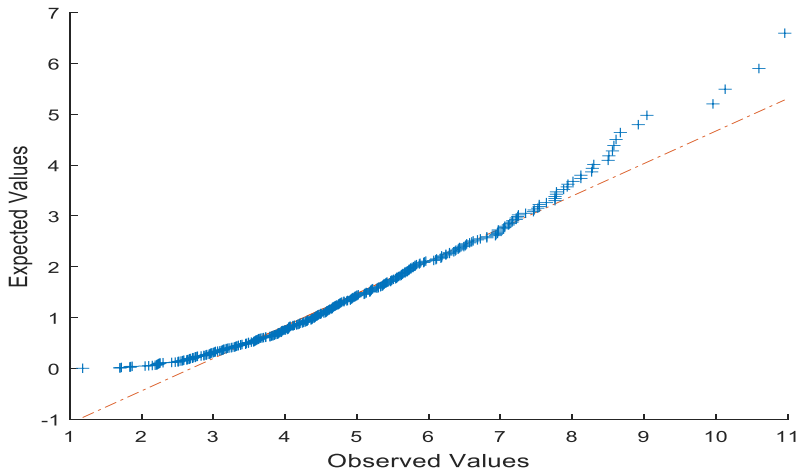


Figure 3. *The EHL q-q plot for wind speed data set*

The ML, LS, WLS, CM, AD, RTAD, LTAD, ADLTS, MPS, MSAD, and MSALD estimates of the parameters of EHL distribution are given in Table 9 together with model selection criteria.

Table 9. *Parameter estimates and model selection criteria values for wind speed data*

	$\hat{\alpha}$	$\hat{\beta}$	KS	RMSE	R^2	AIC	BIC
ML	7.6345	0.6911	0.0284	0.0117	0.9984	1410.7802	1418.5800
LS	7.4522	0.6796	0.0212	0.0092	0.9990	1411.0198	1418.8196
WLS	7.5213	0.6842	0.0240	0.0097	0.9989	1410.8640	1418.6638
CM	7.5484	0.6825	0.0207	0.0092	0.9990	1410.9855	1418.7853
AD	7.4769	0.6825	0.0238	0.0095	0.9989	1410.8938	1418.6935
RTAD	8.2891	0.7035	0.0281	0.0109	0.9986	1411.5785	1419.3783
LTAD	7.0215	0.6658	0.0236	0.0098	0.9988	1411.5732	1419.3730
ADLTS	6.8979	0.6612	0.0246	0.0103	0.9987	1411.8946	1419.6944
MPS	6.4597	0.6465	0.0284	0.0124	0.9980	1413.3581	1421.1579
MSAD	9.7653	0.7562	0.0413	0.0251	0.9931	1416.0738	1423.8736
MSALD	7.3668	0.6846	0.0322	0.0128	0.9980	1410.9008	1418.7006

It can be observed from Table 9 that the CM estimates exhibit the best performance with respect to the cdf-based goodness-of-fit measures, namely the KS statistic, RMSE, and R^2. The CM method is closely followed by the LS, AD, and WLS estimates according to these criteria. On the other hand, when likelihood-based information criteria are considered, the ML estimates yield the smallest AIC and BIC values, indicating a slightly better fit from a likelihood perspective. The differences in AIC and BIC among the competing estimation methods are relatively small, suggesting that the overall fits are comparable. To illustrate these findings, Figure 4 presents the histograms of the fitted pdfs corresponding to the methods that perform well according to the cdf-based measures. As seen in Figure 4, the pdfs based on the CM, AD, and LS estimates provide an adequate fit to the observed data, whereas the pdf based on the MSAD estimates fails to capture the underlying distributional structure.

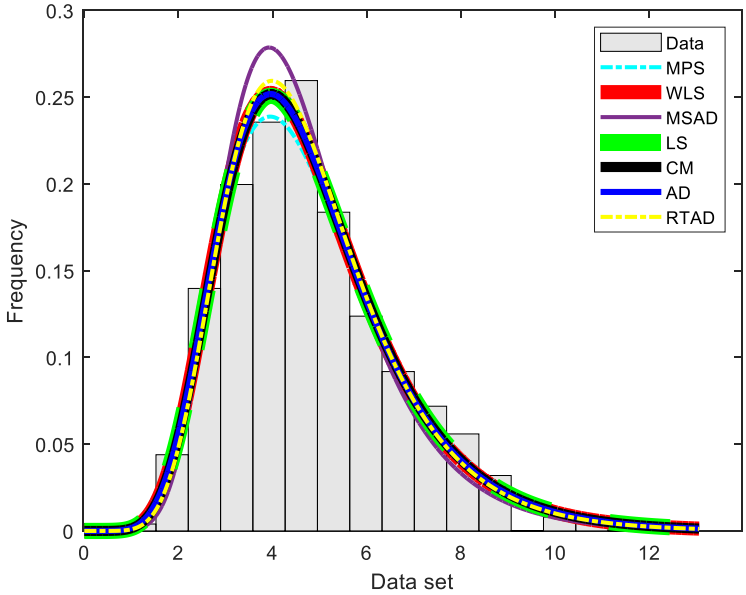


Figure 4. *Histogram of wind speed data with fitted pdfs*

Conclusion

In this paper, the shape and scale parameters of the EHL distribution are estimated using eleven different parameter estimation methods, namely ML, LS, WLS, CM, AD, RTAD, LTAD, ADLTS, MPS, MSAD, and MSALD. A comprehensive Monte Carlo simulation study is carried out to evaluate and compare the performance of these estimators in terms of bias and RMSE criteria with respect to different sample sizes and parameter settings. The simulation results indicate that the MPS, ML, and AD estimators are the most efficient overall, particularly in terms of RMSE, which is consistent with findings reported in the literature for spacing-based and goodness-of-fit–based estimators applied to lifetime distributions. In contrast, the LS and WLS estimators exhibit the smallest bias across most scenarios, especially for small and moderate sample sizes. These results align with previous studies showing that LS–based estimators often provide stable performance in small samples, while likelihood- and spacing-based methods tend to achieve higher efficiency as sample size increases. The practical applicability of the proposed estimation methods is further investigated through a real wind speed data analysis. Based on goodness-of-fit measures and comparative performance criteria, the CM, LS, AD, and WLS estimators provide a better fit to the observed data than the remaining methods. Despite these contributions, the present study has some limitations. The analysis is restricted to the two-parameter EHL distribution and a selected set of classical and distance-based estimation methods. In future studies, we hope to extend this framework to censored data, accelerated life testing schemes, alternative goodness-of-fit measures,

and other extensions of the EHL distribution, as well as to broader applications in environmental and reliability studies.

Declarations

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